

# Quantum Computing over Finite Fields: Reversible Relational Programming with Exclusive Disjunctions

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**Abstract**—In recent work, Benjamin Schumacher and Michael D. Westmoreland investigate a version of quantum mechanics which they call *modal quantum theory* but which we prefer to call *discrete quantum theory*. This theory is obtained by instantiating the mathematical framework of Hilbert spaces with a finite field instead of the field of complex numbers. This instantiation collapses much the structure of actual quantum mechanics but retains several of its distinguishing characteristics including the notions of superposition, interference, and entanglement. Furthermore, discrete quantum theory excludes local hidden variable models, has a no-cloning theorem, and can express natural counterparts of quantum information protocols such as superdense coding and teleportation.

Our first result is to distill a model of discrete quantum computing from this quantum theory. The model is expressed using a monadic metalanguage built on top of a universal reversible language for finite computations, and hence is directly implementable in a language like Haskell. In addition to superpositions and invertible linear maps, the model includes conventional programming constructs including pairs, sums, higher-order functions, and recursion. Our second result is to relate this programming model to relational programming, e.g., a pure version of Prolog over finite relations. Surprisingly discrete quantum computing is identical to conventional logic programming except for a small twist that is responsible for all the “quantum-ness.” The twist occurs when merging sets of answers computed by several alternatives: the answers are combined using an *exclusive* version of logical disjunction. In other words, the two branches of a choice junction exhibit an *interference* effect: an answer is produced from the junction if it occurs in one or the other branch but not both.

## I. THE RESULT, INFORMALLY

Consider this Prolog program:

```
r(false,false).
r(false,true).
r(true,false).

q(X) :- r(false,X).
q(X) :- r(true,X).
```

The program starts with three facts about a relation  $r$  and then defines a rule  $q$  such that  $q(X)$  is true if either of the two

clauses can be satisfied. Executing the query  $q(X)$  returns three answers:

```
x = false ;
x = true  ;
x = false.
```

Now consider the same example expressed in the Discrete Quantum Theory over the field of booleans ( $DQT_2$ ) recently developed by Schumacher and Westmoreland [1]:

$$\begin{aligned} r|0\rangle &= |0\rangle + |1\rangle \\ r|1\rangle &= |0\rangle \end{aligned}$$

$$q = r|0\rangle + r|1\rangle$$

The relation  $r$  which could relate **false** to either **false** or **true** is expressed as a quantum gate  $r$  that maps  $|0\rangle$  to a superposition of  $|0\rangle$  and  $|1\rangle$ . The rule  $q$  corresponds to a vector  $q$  produced by taking the superposition of the two possible alternatives.

In contrast to the Prolog program, measuring the vector  $q$  in the standard basis  $\{|0\rangle, |1\rangle\}$  is guaranteed to *always* return  $|1\rangle$ : the answer  $|0\rangle$  could never be produced! To understand why, we calculate the result of  $q$  as follows:

$$q = r|0\rangle + r|1\rangle = (|0\rangle + |1\rangle) + |0\rangle = |1\rangle$$

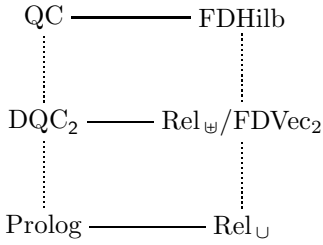
In the last step the two intermediate answers  $|0\rangle$  *interfere* destructively with each other and are canceled. The details are explained in Section IV.

This simple example captures the essence of our result which can be informally stated as follows. *The model of computation inherent in  $DQT_2$ ,  $DQC_2$ , is a relational programming model in which any value that appears an even number of times in disjunctions disappears.* Section VIII-C shows a more significant example that implements the superdense coding quantum protocol in both  $DQC_2$  and Prolog. Removing all answers that appear an even number of times in the Prolog execution is indeed consistent with the quantum protocol.

## II. THE RESULT, FORMALLY

The pioneering research programmes of Abramsky and Coecke et al. [2], [3] and of Selinger [4], [5] have established

that quantum computing (QC) can be modeled using *dagger compact closed categories*. The traditional mathematical model of QC — the category  $\text{FDHilb}$  of finite dimensional Hilbert spaces and linear maps — is a prime example of such categories, *and so is the category  $\text{Rel}_\cup$  of sets and relations*. Our result, formally stated, is that the computational structures inherent in  $\text{DQC}_2$  can be modeled in a non-standard category  $\text{Rel}_\boxplus$  of sets and relations which is isomorphic to the category  $\text{FDVec}_2$  of finite dimensional vector spaces over the field of booleans and linear maps. This category is dagger compact closed and hence possesses all the computational structures necessary for QC. Our category differs from the standard category of sets and relations in one aspect: it uses the *exclusive union* of sets everywhere the standard union would be used. Diagrammatically we have:



### III. SIGNIFICANCE AND BACKGROUND

*Hilbert Spaces and Models of Quantum Computing:* The traditional mathematical formulation of Quantum Mechanics (QM) is founded on Hilbert spaces. Although there are several other more abstract mathematical formulations of QM, the Hilbert space formalism remains the most accepted and the most widely used such formalism [6]. A Hilbert space is defined to be a real or complex inner product space that is a complete metric space with respect to the distance function induced by the inner product. A real or complex inner product space is a vector space over the field of real or complex numbers on which there is an inner product. In traditional QM, the underlying field is the field of complex numbers which serves as the space from which *probability amplitudes* for quantum events are drawn.

For the purposes of QC, the mathematical formalism of QM is typically restricted to finite dimensional Hilbert spaces. This restriction removes one of the infinities in the formalism but retains another — the underlying infinite field of complex numbers. The resulting model is clearly still a mathematical idealization as it allows infinitesimally fine distinctions among quantum states that might differ by vanishingly small probability amplitudes. In contrast, any particular realization of a quantum algorithm can only assume finite and discrete levels of representation of such probability amplitudes. Although it is customary for (classical or quantum) models of computations to include infinite structures of various kinds, it is important to understand how the idealized models emerge as the limits of finite approximations. A fundamental question that therefore motivates our research is whether it is possible to replace the infinite field of complex numbers by successively larger and

larger finite fields to reach full QC in the limit. More importantly, what is the computational power of these intermediate models,  $\text{DQC}_n$ , as  $n$  grows larger?

To tackle these questions, we do not work directly with the Hilbert space formalism, however. Indeed, the mathematical formulation of QC based on Hilbert spaces obscures many traditional computational structures that are inherent in the physical theory. Early on, Mu and Bird [7] showed that the simple model of QC based on vectors and linear operators is a *monad* which is the standard way to model computational effects [8]. In a subsequent development, the third author and collaborators [9], [10], [11], [12], [13] have established that the more general quantum model based on density matrices and superoperators is an instance of the mathematical concept of *arrows* [14] which is a generalization of monads. Interestingly dagger closed compact categories identified by Abramsky et. al. also serve as models of linear logic [15] and various computational effects [16], [17], [18]. Hence, based on these connections, our technical contributions are expressed using traditional computational structures and constructions: monads, arrows, and category theory.

### IV. DISCRETE QUANTUM THEORY

In their recent work, Schumacher and Westmoreland [1] argue that much of the structure of traditional QM is maintained in the presence of finite fields. In particular, they establish that the quantum theory based on the finite field of booleans retains the following characteristics of QM:

- the notions of superposition, interference, entanglement, and mixed states of quantum systems;
- the time evolution of quantum systems using invertible linear operators;
- the complementarity of incompatible observables;
- the exclusion of local hidden variable theories and the impossibility of cloning quantum states; and
- the presence of natural counterparts of quantum information protocols such as superdense coding and teleportation.

*Fields:* A field is an algebraic structure with notions of addition and multiplication that satisfy the usual axioms. The rationals, reals, complex numbers, and quaternions form fields that are infinite. There are also finite fields that satisfy the same set of axioms. Finite fields are necessarily “cyclic.”

We fix a field  $\mathbb{B}$  consisting of two scalars  $\{F, T\}$ . The elements  $F$  and  $T$  are associated with the probabilities of quantum events, with  $F$  interpreted as *definitely no* and  $T$  interpreted as *possibly yes*.<sup>1</sup> The field  $\mathbb{B}$  comes with an addition operation  $\underline{\vee}$  (which in this case must be exclusive-or) and a multiplication operation  $\wedge$  (which in this case must be conjunction). In particular we have:

<sup>1</sup>Everything works if we switch the interpretation with  $F$  interpreted as *possibly no* and  $T$  as *definitely yes* [19].

$$\begin{array}{ll}
F \underline{\vee} F = F & F \wedge F = F \\
F \underline{\vee} T = T & F \wedge T = F \\
T \underline{\vee} F = T & T \wedge F = F \\
T \underline{\vee} T = F & T \wedge T = T
\end{array}$$

The definitions are intuitively consistent with the interpretation of scalars as probabilities for quantum events except that it appears strange to have  $T \underline{\vee} T$  be defined as  $F$ , i.e., to have a twice-possible event become impossible. This results from the cyclic property intrinsic to finite fields requiring the existence of an inverse to  $\underline{\vee}$ . This inverse must be  $T$  itself which means that  $T$  essentially plays both the roles of “possible with phase 1” and “possible with phase -1” and that the two occurrences cancel each other in a superposition.

*Vector Spaces:* In the vector space over the field  $\mathbb{B}$ , if vectors are represented as functions  $\kappa$  that map basis elements  $v$  to scalars field values, then vector addition and scalar multiplication are defined by lifting the field operations to vectors as follows:

$$\begin{array}{ll}
\kappa_1 + \kappa_2 = \kappa' & \text{s.t. } \kappa'(v) = \kappa_1(v) \underline{\vee} \kappa_2(v) \\
b * \kappa = \kappa' & \text{s.t. } \kappa'(v) = b \wedge \kappa(v)
\end{array}$$

Scalar multiplication is uninteresting as it either leaves the vector unchanged or returns the zero vector (denoted  $\bullet$ ) which maps everything to the scalar zero. Vector addition however is the key to introducing superpositions and interference effects.

Consider the simple case of a 1-qubit system with bases  $|0\rangle$  and  $|1\rangle$ . The Hilbert space framework allows us to construct an infinite number of states for the qubit all of the form  $\alpha|0\rangle + \beta|1\rangle$  with  $\alpha$  and  $\beta$  elements of the underlying field of complex numbers and with the side condition that the state is not the zero vector and that it has length 1, i.e., that  $|\alpha|^2 + |\beta|^2 = 1$ . Moving to a finite field immediately limits the set of possible states as the coefficients  $\alpha$  and  $\beta$  are now drawn from a finite set. In other words, in the field  $\mathbb{B}$ , there are exactly three valid states for the qubit:  $F|0\rangle + T|1\rangle$  (which is equivalent to  $|1\rangle$ ),  $T|0\rangle + F|1\rangle$  (which is equivalent to  $|0\rangle$ ), and  $T|0\rangle + T|1\rangle$  (which we write as  $|+\rangle$ ). The fourth possibility is the zero vector which is not an allowed quantum state (discussed below). In a larger field with three scalars, there would be eight possible states for the qubit which intuitively suggests that one must “pay” for the amount of desired superpositions: the larger the finite field, the more states are present with the full Bloch sphere seemingly appearing at the “limit”  $n \rightarrow \infty$ .

Interestingly, we can easily check that the three possible vectors for a 1-qubit state are linearly dependent with any pair of vectors expressing the third as a superposition:

$$\begin{array}{lll}
|0\rangle + |1\rangle & = & |+\rangle \\
|0\rangle + |+\rangle & = & |1\rangle \\
|1\rangle + |+\rangle & = & |0\rangle
\end{array}$$

In other words, other than the standard basis consisting of  $|0\rangle$  and  $|1\rangle$ , there are just two other possible bases for this vector space,  $\{|1\rangle, |+\rangle\}$  and  $\{|+\rangle, |0\rangle\}$ .

The example also shows that the cyclic structure of the field extends to the vector space.

*Inner Products:* A Hilbert space comes equipped with an inner product  $\langle v_1 | v_2 \rangle$  which is an operation that associates each pair of vectors with a complex number scalar value that quantifies the “closeness” of the two vectors. The inner product induces a norm  $\sqrt{\langle v | v \rangle}$  that can be thought of as the length of vector  $v$ . In a finite field, we can still define an operation  $\langle v_1 | v_2 \rangle$  which, following our interpretation of the scalars, would need to return  $F$  if the vectors are definitely not the same and  $T$  if the vectors are possibly the same. This operation however does *not* yield an inner product, as the definition of inner products requires that the field has characteristic 0, i.e., that the repeated addition of  $T$  to itself never reaches  $F$ . As the above paragraph shows this is not the case for  $\mathbb{B}$  (nor for any finite field for that matter) as the sum of positive elements must eventually “wrap around.” In other words, if we choose to instantiate the mathematical framework of Hilbert spaces with a finite field, we must therefore drop the requirement for inner products and content ourselves with a plain vector space. Furthermore, in the absence of a notion of length, one must exclude zero vectors.

*Invertible Linear Maps:* In actual QC, the dynamic evolution of quantum states is described by unitary transformations which preserve inner products. As discrete quantum theory lacks inner products, the dynamic evolution of quantum states is described by any invertible linear transformation, i.e., by any linear transformation that is guaranteed never to produce the zero vector.

As an example, there are 16 linear (not necessarily invertible) functions in the space of 1-qubit functions. Out of these, six are permutations on the three 1-qubit vectors; the remaining all map one of the vectors to  $\bullet$  which makes them non-invertible. Hence the space is quite impoverished compared to the full set of 1-qubit linear transformations in the Hilbert space. In particular, even some of the elementary unitary transformations such as the Hadamard transformation are not expressible in that space. Indeed the Hadamard matrix for the field of booleans maps the vector  $|+\rangle$  to the zero vector and hence is not an acceptable 1-qubit transformation.

*Entanglement and Superdense Coding:* Despite the restriction to finite fields and the drastic reduction in the state space of qubits and their transformations, the theory built on the field of booleans has a definite quantum character. We present the superdense coding example from the paper of Schumacher and Westmoreland [1] in Section VIII-C and refer the reader to their paper for more details.

## V. OUTLINE OF TECHNICAL DEVELOPMENT

Our aims are to (i) develop a typed programming language that pins down the unique features of discrete quantum computation and, (ii) to relate this language to that of conventional relational programming. Building on the work of Abramsky and Coecke [2], [3] and of Selinger [4], [5], both tasks can be achieved by distilling a logic and type system from a special category of sets and relations and by designing an appropriate

syntax. That category (and hence the language) can be related to the conventional category of sets and relations to establish a connection with conventional relational programming and it can be connected to discrete quantum computation by showing that it is dagger compact closed. We proceed in the following two stages.

*Classical Computation:* Many programming models of QC start with the  $\lambda$ -calculus as the underlying classical language and add quantum features on top of it [20], [21], [22], [23]. This strategy is natural given that the  $\lambda$ -calculus is the canonical classical computational model. However this strategy complicates the development of quantum languages as it forces the languages to deal in fairly complicated ways with the implicit duplication and erasure of information in the classical sublanguage. A simpler strategy that loses no generality is to build the quantum features on top of a reversible classical language: this keeps the language simple while still enabling the  $\lambda$ -calculus constructs to be encoded using two explicit operators for erasure and duplication if desired [24]. Following this strategy we review the reversible language  $\Pi$  that was recently developed by the first and third author and use it as a foundation for the quantum language in the remainder of the paper. The language provides the symmetrical monoidal part of the required categorical structure.

*Monads and Arrows for Quantum Computation:* We define a language  $\mathcal{R}\Pi$  by adding a layer on top of the classical reversible language  $\Pi$  that provides sets and relations. The set construction can be expressed as a *strong monad* [25] over the language of classical observable values. However in order to express the required compact closure structure in the language, the Kleisli maps of this monad are abstracted in a first-order data type using an arrow [14]. The language  $\mathcal{R}\Pi$  is expressive enough to implement the superdense coding protocol and other quantum protocols. It differs from a conventional relational programming language in the semantics of the “union” operation.

## VI. CLASSICAL COMPUTATION

We introduce the language  $\Pi$  for classical observable values. The language is defined over finite types: the only “computations” in it are the isomorphisms between these types. In the context of Hilbert spaces, this language formalizes the notation used for the bases. In the categorical context, this language provides the underlying symmetric monoidal structure.

### A. Bases for Vector Spaces

In the traditional presentation of quantum computing, a Hilbert space of dimension  $d$  is spanned by a set of  $d$  mutually orthogonal vectors. For example:

- a quantum system  $Q_2$  composed of just 1 qubit is of dimension 2 and is spanned by  $\{|0\rangle, |1\rangle\}$ ;
- a quantum system  $Q_4$  composed of 2 qubits is of dimension 4 and is spanned by  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ .

The Hilbert space for the composition of two systems is spanned by basis vectors of the form  $|m \cdot n\rangle$  where  $m$  is a basis vector of the first system and  $n$  is a basis vector of the

second system and the operation  $(\cdot)$  is the composition of the two labels. For example:

- the composition of  $Q_2$  and  $Q_4$  (written  $Q_2 \otimes Q_4$ ) is of dimension 8 and is spanned by:  
 $\{|0\cdot00\rangle, |0\cdot01\rangle, |0\cdot10\rangle, |0\cdot11\rangle, |1\cdot00\rangle, |1\cdot01\rangle, |1\cdot10\rangle, |1\cdot11\rangle\}$
- the composition of  $Q_4$  and  $Q_2$  (written  $Q_4 \otimes Q_2$ ) is also of dimension 8 and is spanned by:  
 $\{|00\cdot0\rangle, |00\cdot1\rangle, |01\cdot0\rangle, |01\cdot1\rangle, |10\cdot0\rangle, |10\cdot1\rangle, |11\cdot0\rangle, |11\cdot1\rangle\}$

Clearly the compositions  $Q_2 \otimes Q_4$  and  $Q_4 \otimes Q_2$  are isomorphic and the distinction between them is typically blurred.

### B. Syntax and Type System

Instead of using bits as the labels for the bases, we use structured finite types built using the empty type, the unit type, the sum type, and the product type. Furthermore instead of silently identifying systems like  $Q_2 \otimes Q_4$  and  $Q_4 \otimes Q_2$ , we include an explicit set of operators between these finite types to witness the isomorphisms. Specifically, we have the following language of classical observable types  $b$  and values  $v$ :

$$\begin{aligned} b &::= 0 \mid 1 \mid b + b \mid b \times b \mid \text{bool} \\ v &::= () \mid \text{left } v \mid \text{right } v \mid (v, v) \mid \text{T} \mid \text{F} \end{aligned}$$

The types are finite as they will be used to define the dimensions of our vector spaces. The type 0 is the empty type containing no inhabitants. The type 1 has exactly one inhabitant called  $()$ . The type  $b_1 + b_2$  is the disjoint union of  $b_1$  and  $b_2$  whose elements are appropriately tagged values from either type. The type  $b_1 \times b_2$  is the type of ordered pairs whose elements are coming from  $b_1$  and  $b_2$  respectively. Although the type of booleans is expressible as  $1 + 1$  we add it as a primitive type because it corresponds to the type of scalar field values which play an important role in the language. The type system is summarized below:

$$\begin{array}{c} \frac{}{\vdash () : 1} \quad \frac{\vdash v : b_1}{\vdash \text{left } v : b_1 + b_2} \quad \frac{\vdash v : b_2}{\vdash \text{right } v : b_1 + b_2} \\ \frac{\vdash v_1 : b_1 \quad \vdash v_2 : b_2}{\vdash (v_1, v_2) : b_1 \times b_2} \quad \frac{}{\vdash \text{T} : \text{bool}} \quad \frac{}{\vdash \text{F} : \text{bool}} \end{array}$$

The following isomorphisms are sound and complete for the given finite types [26]:

$$\begin{aligned} b &\leftrightarrow b \\ 0 + b &\leftrightarrow b \\ b_1 + b_2 &\leftrightarrow b_2 + b_1 \\ b_1 + (b_2 + b_3) &\leftrightarrow (b_1 + b_2) + b_3 \\ 1 \times b &\leftrightarrow b \\ b_1 \times b_2 &\leftrightarrow b_2 \times b_1 \\ b_1 \times (b_2 \times b_3) &\leftrightarrow (b_1 \times b_2) \times b_3 \\ 0 \times b &\leftrightarrow 0 \\ (b_1 + b_2) \times b_3 &\leftrightarrow (b_1 \times b_3) + (b_2 \times b_3) \\ \text{bool} &\leftrightarrow 1 + 1 \end{aligned}$$

$$\frac{b_1 \leftrightarrow b_2 \quad b_2 \leftrightarrow b_3}{b_1 \leftrightarrow b_3}$$

$$\frac{b_1 \leftrightarrow b_3 \quad b_2 \leftrightarrow b_4}{(b_1 + b_2) \leftrightarrow (b_3 + b_4)} \quad \frac{b_1 \leftrightarrow b_3 \quad b_2 \leftrightarrow b_4}{(b_1 \times b_2) \leftrightarrow (b_3 \times b_4)}$$

We introduce primitive operators corresponding to the left-to-right and right-to-left reading of each isomorphism. We gather these operators into the table below.

$id :$	$b \leftrightarrow b$	$: id$
$\succ_+ :$	$0 + b \leftrightarrow b$	$: \preccurlyeq_+$
$\times_+ :$	$b_1 + b_2 \leftrightarrow b_2 + b_1$	$: \times_+$
$\geq_+ :$	$b_1 + (b_2 + b_3) \leftrightarrow (b_1 + b_2) + b_3$	$: \leq_+$
$\succ_\times :$	$1 \times b \leftrightarrow b$	$: \preccurlyeq_\times$
$\times_\times :$	$b_1 \times b_2 \leftrightarrow b_2 \times b_1$	$: \times_\times$
$\geq_\times :$	$b_1 \times (b_2 \times b_3) \leftrightarrow (b_1 \times b_2) \times b_3$	$: \leq_\times$
$<_0 :$	$0 \times b \leftrightarrow 0$	$: >_0$
$< :$	$(b_1 + b_2) \times b_3 \leftrightarrow (b_1 \times b_3) + (b_2 \times b_3)$	$: >$
$\in :$	$bool \leftrightarrow 1 + 1$	$: \oplus$

Each line of this table is to be read as the definition of two operators. For example corresponding to the identity of  $\times$  isomorphism we have the two operators  $\succ_\times : 1 \times b \leftrightarrow b$  and  $\preccurlyeq_\times : b \leftrightarrow 1 \times b$ .

Now that we have primitive operators we need some means of composing them. We construct the composition combinators out of the closure conditions for isomorphisms. Thus we have one sequential composition combinator  $\circ$  and two parallel composition combinators, one for sums  $\oplus$  and one for pairs  $\otimes$ .

$$\frac{c_1 : b_1 \leftrightarrow b_2 \quad c_2 : b_2 \leftrightarrow b_3}{(c_1 \circ c_2) : b_1 \leftrightarrow b_3} \quad \frac{c_1 : b_1 \leftrightarrow b_3 \quad c_2 : b_2 \leftrightarrow b_4}{(c_1 \oplus c_2) : (b_1 + b_2) \leftrightarrow (b_3 + b_4)}$$

$$\frac{c_1 : b_1 \leftrightarrow b_3 \quad c_2 : b_2 \leftrightarrow b_4}{(c_1 \otimes c_2) : (b_1 \times b_2) \leftrightarrow (b_3 \times b_4)}$$

**Definition 6.1:** (Syntax of  $\Pi$ ) We collect our combinators, types and values to get the definition of our language for isomorphisms, which we will refer to as  $\Pi$ :

$$\begin{aligned} b &::= 0 \mid 1 \mid b + b \mid b \times b \mid bool \\ v &::= () \mid left \ v \mid right \ v \mid (v, v) \mid F \mid T \\ t &::= b \leftrightarrow b \\ iso &::= \times_+ \mid \geq_+ \mid \leq_+ \mid \succ_+ \mid \preccurlyeq_+ \\ &\quad \mid \times_\times \mid \geq_\times \mid \leq_\times \mid \succ_\times \mid \preccurlyeq_\times \\ &\quad \mid <_0 \mid >_0 \mid < \mid > \mid id \mid \in \mid \oplus \\ c &::= iso \mid c \circ c \mid c \otimes c \mid c \oplus c \end{aligned}$$

Given a program  $c : b_1 \leftrightarrow b_2$  in  $\Pi$ , we can run it either in the forward direction by applying it to a value  $v_1 : b_1$  or run it in the opposite direction by applying it to a value  $v_2 : b_2$ .

The forward  $c \ v_1 \mapsto v_2$  transitions are given below:

$id$	$v$	$\mapsto v$
$\succ_+$	$(right \ v)$	$\mapsto v$
$\preccurlyeq_+$	$v$	$\mapsto right \ v$
$\times_+$	$(left \ v)$	$\mapsto right \ v$
$\times_+$	$(right \ v)$	$\mapsto left \ v$
$\geq_+$	$(left \ v_1)$	$\mapsto left \ (left \ v_1)$
$\geq_+$	$(right \ (left \ v_2))$	$\mapsto left \ (right \ v_2)$
$\geq_+$	$(right \ (right \ v_3))$	$\mapsto right \ v_3$
$\leq_+$	$(left \ (left \ v_1))$	$\mapsto left \ v_1$
$\leq_+$	$(left \ (right \ v_2))$	$\mapsto right \ (left \ v_2)$
$\leq_+$	$(right \ v_3)$	$\mapsto right \ (right \ v_3)$
$\succ_\times$	$((), v)$	$\mapsto v$
$\preccurlyeq_\times$	$v$	$\mapsto ((), v)$
$\times_\times$	$(v_1, v_2)$	$\mapsto (v_2, v_1)$
$\geq_\times$	$(v_1, (v_2, v_3))$	$\mapsto ((v_1, v_2), v_3)$
$\leq_\times$	$((v_1, v_2), v_3)$	$\mapsto (v_1, (v_2, v_3))$
$<$	$(left \ v_1, v_3)$	$\mapsto left \ (v_1, v_3)$
$<$	$(right \ v_2, v_3)$	$\mapsto right \ (v_2, v_3)$
$>$	$(left \ (v_1, v_3))$	$\mapsto (left \ v_1, v_3)$
$>$	$(right \ (v_2, v_3))$	$\mapsto (right \ v_2, v_3)$
$\in$	$T$	$\mapsto left \ ()$
$\in$	$F$	$\mapsto right \ ()$
$\oplus$	$(left \ ())$	$\mapsto T$
$\oplus$	$(right \ ())$	$\mapsto F$

Since there are no values that have the type 0, the reductions for the combinators  $\succ_+$ ,  $\preccurlyeq_+$ ,  $<_0$  and  $>_0$  omit the impossible cases. The semantics of the other combinators is straightforward. Composition combinators are defined as follows:

$$\frac{c_1 \ v_1 \mapsto v_2}{(c_1 \oplus c_2) \ (left \ v_1) \mapsto left \ v_2} \quad \frac{c_2 \ v_1 \mapsto v_2}{(c_1 \oplus c_2) \ (right \ v_1) \mapsto right \ v_2}$$

$$\frac{c_1 \ v_1 \mapsto v_3 \quad c_2 \ v_2 \mapsto v_4}{(c_1 \otimes c_2) \ (v_1, v_2) \mapsto (v_3, v_4)} \quad \frac{c_1 \ v_1 \mapsto v \quad c_2 \ v \mapsto v_2}{(c_1 \circ c_2) \ v_1 \mapsto v_2}$$

For the inverse direction, it is a simple matter to establish the following property.

**Proposition 6.2:** For every  $\Pi$  program  $c$ , such that  $c \ v \mapsto v'$ , we can construct its adjoint  $c^\dagger$  such that  $c^\dagger v' \mapsto v$ .

*Proof:* We can construct the required  $c^\dagger$  by replacing every primitive isomorphism with its dual. For sequential composition we have  $(c_1 \circ c_2)^\dagger = c_2^\dagger \circ c_1^\dagger$  and for parallel composition we have  $(c_1 \otimes c_2)^\dagger = c_1^\dagger \otimes c_2^\dagger$  and  $(c_1 \oplus c_2)^\dagger = c_1^\dagger \oplus c_2^\dagger$ . ■

### C. Dagger Symmetrical Monoidal Structure

The language  $\Pi$  is rich enough to express the required dagger symmetric monoidal (but not the compact close structure) structure. The proof for this is straightforward and we content ourselves with a brief outline. The language  $\Pi$  can be interpreted as category whose objects are the base types  $b$  and whose morphisms the combinators  $c$ . Identity morphisms are given by  $id$  and associativity follows from the composition of

isomorphisms, thus establishing the required properties for a category.

Further  $\Pi$  is a monoidal category  $(b, \times, 1)$  where  $\times$  is the tensor operation,  $1$  is the identity object and the required natural transforms  $\alpha$ ,  $\lambda$  and  $\rho$  are given by  $\leq_{\times}$ ,  $\succ_{\times}$  and  $\times_{\times} \circ \succ_{\times}$  respectively. The required “pentagon” and “triangle” axioms follow from the definitions of these isomorphisms.

Braiding is provided by  $\times_{\times}$  at the appropriate types and it satisfies the “hexagon” axioms. Further that  $\times_{\times}$  at  $b_1 \times b_2$  is the same as  $\times_{\times}^{-1}$  at  $b_2 \times b_1$  establishing symmetry.

Finally, proposition 6.2 tells us that for every morphism  $c$  there exists its adjoint  $c^{\dagger}$  establishing that  $\Pi$  is a dagger symmetric monoidal category.

## VII. THE LANGUAGE $\mathcal{R}\Pi$

The core language for discrete quantum computing is obtained by adding a layer on top of the reversible core presented in the previous section. The additional layer is that of vectors and linear maps or equivalently that of sets and relations.

*Definition 7.1:* (Syntax of Core  $\mathcal{R}\Pi$ )

$$\begin{aligned} b &::= \dots \mid S b \mid b R b \\ v &::= \dots \mid s \\ s &::= \emptyset \mid \{v\} \mid s \uplus s \\ r &::= \text{arr iso} \mid r \ggg r \mid \text{second } r \\ &\quad \mid \text{strength} \mid \text{state } s \mid \eta_b \mid \varepsilon \end{aligned}$$

The language  $\mathcal{R}\Pi$  extends  $\Pi$  as follows. The set of types is extended with the type  $S b$  of sets of values and the type  $b_1 R b_2$  of relations between sets of values of type  $b_1$  and sets of values of type  $b_2$ . The set of values is extended with sets  $s$  which can either be the empty set  $\emptyset$ , a singleton set  $\{v\}$ , or the *exclusive union*  $s_1 \uplus s_2$ . The exclusive union of sets  $s_1$  and  $s_2$  is the union of all the elements that are not in common. This union reflects the cyclic nature of the underlying finite field as was explained in Section IV. In more detail, we saw for example that adding the vector  $F|0\rangle + T|1\rangle$  to the vector  $T|0\rangle + T|1\rangle$  produces the vector  $T|0\rangle + F|1\rangle$  where the two occurrences of the component  $|1\rangle$  canceled each other. Expressed as sets, we would have that  $\{1\} \uplus \{0, 1\}$  is equal to  $\{0\}$ .

The layer of sets corresponds to a *strong monad* with the singleton set as the unit and the folding of the exclusive union as the bind operation. The Kleisli arrows of this monad are functions that map values to sets. In order to express these functions themselves as relations, we provide explicit arrow operators to construct them. The language therefore includes a separate syntactic category of relations which is used to build the required Kleisli maps using arrow primitives. The operator *arr* lifts any isomorphism from the underlying reversible language to a relation on sets. The operator  $\ggg$  sequences two relations and the operator *second* applies a relation to the second component of a pair leaving the first component unchanged. These three operators are the minimal core operators for any arrow language. In addition, we must include enough structure to provide a compact closed category: the operator *strength* is used to express the tensor product of

sets; the operator *state* identifies a state of type  $S b$  with an arrow (relation) of type  $1 R b$  [3]; finally the pair of operators  $\eta_b$  and  $\varepsilon$  are the unit elimination and introduction that are required for any compact closed category. The operation  $\eta$  is indexed by a type  $b$ .

We present the type system below. The remainder of this section explains the semantics of core  $\mathcal{R}\Pi$  formally and illustrates its use in several examples:

$$\begin{aligned} &\frac{}{\vdash \emptyset : S b} \quad \frac{}{\vdash \{v\} : S b} \quad \frac{}{\vdash s_1 : S b \quad \vdash s_2 : S b} \quad \frac{}{\vdash s_1 \uplus s_2 : S b} \\ &\frac{}{\vdash \text{iso} : b \leftrightarrow b} \quad \frac{}{\vdash \text{arr iso} : b R b} \quad \frac{}{\vdash r_1 : b_1 R b_2 \quad \vdash r_2 : b_2 R b_3} \quad \frac{}{\vdash r_1 \ggg r_2 : b_1 R b_3} \\ &\frac{}{\vdash r : b_1 R b_2} \quad \frac{}{\vdash \text{second } r : (b \times b_1) R (b \times b_2)} \quad \frac{}{\vdash s : S b} \quad \frac{}{\vdash \text{state } s : 1 R b} \\ &\frac{}{\vdash \text{strength} : (b_1 \times S b_2) R (b_1 \times b_2)} \\ &\frac{}{\vdash \eta_b : 1 R (b \times b)} \quad \frac{}{\vdash \varepsilon : (b \times b) R 1} \end{aligned}$$

### A. Semantics

A program in  $\mathcal{R}\Pi$  consists of the application of a relation  $r$  to a set  $s$  producing a resulting set  $s'$ . We model this evaluation using two applications: the application  $r \overline{\text{@}} s$  applies the relation  $r$  to the set  $s$  and the application  $r @ v$  applies the relation  $r$  to an individual set element  $v$ . The first application  $\overline{\text{@}}$  simply iterates down the structure of the set until it finds individual elements which can be processed using  $@$ :

$$\begin{aligned} r \overline{\text{@}} \emptyset &\mapsto \emptyset \\ r \overline{\text{@}} \{v\} &\mapsto r @ v \\ r \overline{\text{@}} (s_1 \uplus s_2) &\mapsto (r \overline{\text{@}} s_1) \uplus (r \overline{\text{@}} s_2) \end{aligned}$$

The application  $@$  applies a relation to a single value  $v$  and returns the set of possible values that are related to  $v$ :

$$\begin{aligned} (\text{arr iso}) @ v &\mapsto \{iso(v)\} \\ (r_1 \ggg r_2) @ v &\mapsto r_2 \overline{\text{@}} (r_1 @ v) \\ (\text{second } r) @ (v_1, v_2) &\mapsto \text{strength} @ (v_1, r @ v_2) \\ \text{strength} @ (v, \emptyset) &\mapsto \emptyset \\ \text{strength} @ (v_1, \{v_2\}) &\mapsto \{(v_1, v_2)\} \\ \text{strength} @ (v, s_1 \uplus s_2) &\mapsto \text{strength} @ (v, s_1) \uplus \text{strength} @ (v, s_2) \\ (\text{state } s) @ () &\mapsto s \\ \eta_b @ () &\mapsto \biguplus_i \{(v_i, v_i)\}, \text{ where } v_i \in b \\ \varepsilon @ (v, v) &\mapsto \{()\} \\ \varepsilon @ (v, v') &\mapsto \emptyset \text{ if } v \neq v' \end{aligned}$$

When applied to a value  $v$ , the relation *arr iso* simply applies the underlying isomorphism to the value which is wrapped in a singleton set. Applying the sequence  $r_1$  and  $r_2$  to a value  $v$  applies  $r_1$  to  $v$  to produce a set which is passed to  $r_2$ . To apply a relation  $r$  to the second component of a pair  $(v_1, v_2)$ , we first produce  $(v_1, r @ v_2)$  and then use the tensorial strength of the monad to push the pair construction through the resulting set as shown in the three rules for *strength*. The relation *state*  $s$  maps the unit value  $()$  to the given set  $s$ . When applied to the unit value  $()$ , the relation  $\eta_b$

produces a superposition of all pairs of values  $v_i$  where  $v_i$  is an element of the type  $b$  (which is finite). The relation  $\varepsilon$  maps a pair of equal values to the singleton set containing  $()$  and maps a pair of different values to the empty set. As the semantics for  $\eta_b$  and  $\varepsilon$  shows, these constructs exploit the fact that the underlying language of classical values is based on finite values (and hence values that can be compared for equality and enumerated). In order to accommodate sets whose elements are themselves sets or relations, it is necessary to have a first-order representation of relations using the arrow constructors instead of using the higher-order Kleisli maps of the monad.

It is a routine task to verify that the evaluation rules preserve the types.

**Proposition 7.2:** If  $\vdash r : b_1 R b_2$  and  $\vdash s : S b_1$  then  $\vdash r @ s : S b_2$ .

### B. Derived Relations

The language  $\mathcal{R}\Pi$  is surprisingly expressive: using standard categorical constructions, it can express higher-order functions, currying, uncurrying, recursion, adjoints, dot products, and outer products. We illustrate these derived relations. The superdense coding protocol in Section VIII-C uses the  $s_2r$  construction below.

We start by defining for every relation  $r$  a relation  $first\ r$  that applies  $r$  to the first component of a pair:

$$\begin{aligned} first & : a R b \rightarrow (a \times c) R (b \times c) \\ first\ r & = arr \times_\times \ggg second\ r \ggg arr \times_\times \end{aligned}$$

Next we present currying and uncurrying and use them to turn any set of pairs to a relation:

$$\begin{aligned} curry & : ((c \times a) R b) \rightarrow (c R (a \times b)) \\ curry\ r & = arr (\preceq_\times \circ \times_\times) \ggg second\ \eta_a \ggg \\ & \quad arr \succeq_\times \ggg first\ r \ggg arr \times_\times \\ uncurry & : (c R (a \times b)) \rightarrow ((c \times a) R b) \\ uncurry\ r & = first\ r \ggg arr (\times_\times \circ \succeq_\times) \ggg \\ & \quad \varepsilon \ggg \preceq_\times \end{aligned}$$

$$\begin{aligned} s_2r & : S (a, b) \rightarrow (a R b) \\ s_2r\ s & = arr \preceq_\times \ggg uncurry (state\ s) \end{aligned}$$

We can also define a *trace* operation to model recursion from cyclic sharing [17]:

$$\begin{aligned} trace & : ((a \times c) R (b \times c)) \rightarrow (a R b) \\ trace\ r & = arr \preceq_\times \ggg first\ \eta_c \ggg \\ & \quad arr (\times_\times \circ \succeq_\times) \ggg first\ r \ggg arr \preceq_\times \\ & \quad \ggg second\ \varepsilon \ggg arr (\times_\times \circ \preceq_\times) \end{aligned}$$

Finally, we can also define an adjoint for every relation and use it to define a *costate* relation that matches a given state. We can combine a state and a costate in two different ways to simulate the dot product and the outer product

constructions [3].

$$\begin{aligned} adjoint & : (a R b) \rightarrow (b R a) \\ adjoint\ r & = arr \preceq_\times \ggg first\ \eta_a \ggg \\ & \quad first\ (second\ r) \ggg arr \preceq_\times \ggg \\ & \quad second\ \varepsilon \ggg arr (\times_\times \circ \preceq_\times) \end{aligned}$$

$$\begin{aligned} costate & : S a \rightarrow (a R 1) \\ costate\ s & = adjoint (state\ s) \end{aligned}$$

$$\begin{aligned} \langle \cdot | \cdot \rangle & : S a \rightarrow S a \rightarrow (1 R 1) \\ \langle s_1 | s_2 \rangle & = state\ s_1 \ggg costate\ s_2 \end{aligned}$$

$$\begin{aligned} |\cdot\rangle\langle\cdot| & : S a \rightarrow S a \rightarrow (a R a) \\ |s_1\rangle\langle s_2| & = costate\ s_2 \ggg state\ s_1 \end{aligned}$$

The result of the dot product is a relation of type  $1 R 1$  which corresponds to a scalar [3].

### C. Interpretations

There are two isomorphic ways of thinking about  $\mathcal{R}\Pi$ : a value of type  $S\ b$  could be viewed as a set of values of type  $b$  or as a vector which maps values of type  $b$  to scalars in the field of booleans. For example,  $s : S (bool \times bool) = \{(F, F)\} \uplus \{(T, T)\}$  denotes the set  $\{(F, F), (T, T)\}$  in the first interpretation and denotes the vector:

$$\begin{pmatrix} (F, F) \\ (F, T) \\ (T, F) \\ (T, T) \end{pmatrix} \begin{pmatrix} T \\ F \\ F \\ T \end{pmatrix}$$

in the second interpretation. In the vector notation, the labels on the left enumerate all the values of the given type. A value is present if the corresponding entry in the vector is T and absent if the corresponding entry is F. To avoid clutter we will assume a fixed preferred ordering of the labels on the left and omit them.

Similarly a value of type  $b_1 R b_2$  can be viewed as a relation mapping sets of values of type  $b_1$  to sets of values of type  $b_2$  or as a linear map which given our preferred ordering could be represented as a matrix. For example, the values:

$$\begin{aligned} r_1, r_2 & : bool R bool \\ r_1 & = s_2r (\{(F, F)\} \uplus \{(T, F)\} \uplus \{(T, T)\}) \\ r_2 & = s_2r (\{(F, F)\} \uplus \{(F, T)\} \uplus \{(T, F)\}) \end{aligned}$$

denote the relations  $\{(F, F), (T, F), (T, T)\}$  and  $\{(F, F), (F, T), (T, F)\}$  in the first interpretation, and the matrices:

$$r_1 = \begin{pmatrix} T & T \\ F & T \end{pmatrix} \quad r_2 = \begin{pmatrix} T & T \\ T & F \end{pmatrix}$$

in the second interpretation. In the matrix notation, the columns are implicitly indexed with F and T from left to right, and the rows are implicitly indexed with F and T from top to bottom. An entry is T if the pair (column-label, row-label) is in the relation and F otherwise.

In a conventional setting, the composition of  $r_1$  and  $r_2$  would be:

$$r_1 \gg r_2 = \{(F, F), (F, T), (T, F), (T, T)\}$$

This is the semantics one gets in conventional relational programming, i.e., pure Prolog and backtracking monads [27], [28] which indeed can be implemented and formalized using sequences or sets [29].

However in  $\mathcal{RII}$ , the composition of these two same relations is  $\{(F, F), (F, T), (T, T)\}$  because the pair  $(T, F)$  can be produced in two different ways which cancel due to interference. It is perhaps more intuitive to compute the composition in the world of matrices:

$$r_1 \gg r_2 = \begin{pmatrix} T & T \\ T & F \end{pmatrix} \begin{pmatrix} T & T \\ F & T \end{pmatrix} = \begin{pmatrix} T \vee F & T \vee T \\ T \vee F & T \vee F \end{pmatrix} = \begin{pmatrix} T & F \\ T & T \end{pmatrix}$$

We can give similar interpretations for each of the  $\mathcal{RII}$  constructs. In the world of matrices, *second*  $r$  denotes the matrix produced by the tensor product of the identity matrix and the matrix corresponding to  $r$ . The construct *state* is a no-op: it allows us to view a vector of size  $n$  as a matrix of dimensions  $n \times 1$ . We illustrate the matrix denoted by *strength* at the type  $(bool \times S \ bool) \ R \ (bool \times bool)$

$$\begin{pmatrix} F & T & F & T & F & F & F & F \\ F & F & T & T & F & F & F & F \\ F & F & F & F & F & T & F & T \\ F & F & F & F & F & F & T & T \end{pmatrix}$$

The rows are indexed from top to bottom by  $(F, F)$ ,  $(F, T)$ ,  $(T, F)$ , and  $(T, T)$ . The columns are indexed from left to right by  $(F, \emptyset)$ ,  $(F, \{F\})$ ,  $(F, \{T\})$ ,  $(F, \{F\} \uplus \{T\})$ ,  $(T, \emptyset)$ ,  $(T, \{F\})$ ,  $(T, \{T\})$ , and  $(T, \{F\} \uplus \{T\})$ .

The matrices corresponding to  $\eta_b$  and  $\varepsilon$  are column and row vectors with entries T at the diagonal elements. For example,  $\eta_{bool} : 1 \ R \ (bool \times bool)$  and  $\varepsilon : (bool \times bool) \ R \ 1$  are:

$$\eta = \begin{pmatrix} T \\ F \\ F \\ T \end{pmatrix} \quad \varepsilon = \begin{pmatrix} T & F & F & T \end{pmatrix}$$

As a special case, we note that relations of type  $1 \ R \ 1$  denote matrices of dimension  $1 \times 1$ , i.e., scalars. We interpret the matrix  $(T)$  as the scalar T and the matrix  $(F)$  as the scalar F.

**Lemma 7.3:** The semantics of  $\mathcal{RII}$  is *sound* with respect to both the relational interpretation and the vector interpretation.

#### D. Dagger Compact Closed Structure

We can now verify that the arrow fragment of the language  $\mathcal{RII}$  forms a dagger compact closed category. The objects of the category are the types  $S \ b$  and the morphisms are the relations  $b_1 \ R \ b_2$ . These form a category since we have identity morphisms due to *arr id* and the composition of relations is associative due to the associativity of the matrix multiplication in the model (soundness). The category has the required monoidal structure with  $S \ (b_1 \times b_2)$  being the product for the objects  $S \ b_1$  and  $S \ b_2$  and  $S \ 1$  is the identity object

for tensors and the required natural transforms  $\alpha$ ,  $\lambda$  and  $\rho$  are *arr* lifted forms of their  $\Pi$  equivalents. Similarly braiding and symmetry are preserved by lifting  $\times_\times$  from  $\Pi$ .

The dagger structure on relations is given by the contravariant functor *adjoint* :  $b_1 \ R \ b_2 \rightarrow b_2 \ R \ b_1$  which we constructed in Section VII-B and is easily verified to be involutive.

Finally the required autonomous structure is given by choosing the dual of  $S \ b$  to be the object  $S \ b$  itself. The required *unit* and *counit* are given by  $\eta$  and  $\varepsilon$  in  $\mathcal{RII}$ . The required adjunction triangles become the same as the right and left duals on objects coincide. Thus the arrow fragment of  $\mathcal{RII}$  forms a dagger compact closed category.

### VIII. USING $\mathcal{RII}$ FOR QUANTUM COMPUTATION

Two more things are needed to use  $\mathcal{RII}$  for quantum computation. First, the zero vector (which is denoted by  $\emptyset$ ) should never be encountered during computation as it does not correspond to a valid quantum state. Second, we must define a notion of measurement.

#### A. Evolution of Quantum States

As Schumacher and Westmoreland [1] explain, the evolution of quantum states in  $DQC_2$  only requires that the linear maps be invertible. We have already established that every linear map has an adjoint but these adjoints do not necessarily coincide with the inverses. Consider the matrix:

$$\begin{pmatrix} T & T \\ T & T \end{pmatrix}$$

The adjoint of this matrix is itself but when multiplied by itself in the field of booleans it produces the zero matrix. In other words, any  $\mathcal{RII}$  expression that can denote this matrix should not be allowed. All the expressions in the syntactic category  $r$  in  $\mathcal{RII}$  denote invertible matrices except for  $\varepsilon$ . This suggests a possibility for tracking the uses of  $\varepsilon$  using an extended type system to syntactically guarantee that a given expression denotes an invertible transformation.

#### B. Measurement

In standard QM, the postulate of measurement requires the notion of an *orthogonal projection*. However, as explained in Section IV, discrete quantum theories lack inner products and hence lack a standard notion of orthogonality. Nevertheless it is possible to define a sensible notion of measurement as explained by Schumacher and Westmoreland [1]. The key technical observations are (i) any set of linearly independent vectors can form a basis, and (ii) it is possible to define dual vectors for a given basis.

In more detail, assuming the vector interpretation of  $\mathcal{RII}$ , a measurement (or an observable) corresponds to a basis  $\{s_1, s_2, \dots\}$  where the collection of values  $s_i$  denote linearly independent vectors. As discussed in Section IV, the space of 1-qubit vectors contains just three vectors with any pair forming a basis. For each choice of basis, we associate an



observable:

$$\begin{aligned} X\text{-basis} &= \{\{T\}, \{F\} \uplus \{T\}\} \\ Y\text{-basis} &= \{\{F\} \uplus \{T\}, \{F\}\} \\ Z\text{-basis} &= \{\{F\}, \{T\}\} \end{aligned}$$

We then associate a *basis dependent* dual  $\bar{s}_i$  to each vector  $s_i$  such that  $\langle \bar{s}_i \mid s_j \rangle$  denotes the scalar  $T$  if and only if  $i = j$ . For example, the dual vectors for the vectors in the  $X$ -basis are:

$$X\text{-dual}_X = \{\{F\} \uplus \{T\}, \{F\}\}$$

The simplest way then to extend  $\mathcal{R}II$  is to add one construct *measure* that takes two arguments: a vector and a set of dual basis vectors along which the vector should be measured. For example, to measure the vector  $\{T\}$  in the  $X$ -basis, we would write: *measure*  $\{T\}\{\{F\} \uplus \{T\}, \{F\}\}$ . The result of measurement is any dual basis vector that can possibly match the given vector selected at random. In the above example, only the first dual basis vector can possibly match the given vector which means that the result is deterministic.

A remarkable feature of the categorical approach to QM is that the semantics of measurement can be expressed as follows in  $\mathcal{R}II$ :

$$\text{measure } s \ \{d_1, d_2, \dots\} = d_i \text{ if } \langle d_i \mid s \rangle \text{ denotes } (T)$$

As explained in Sections VII-B and VII-C, the dot product is expressible in  $\mathcal{R}II$  and the relations of type 1  $R$  1 are identified with scalars. If more than one dual vector produces the scalar  $T$  then the result of measurement is non-deterministic with the dual vector  $d_i$  picked at random.

It should also be possible to exploit the fact that our categories have biproducts to extend the language in a richer way by allowing measurements to occur in the middle of computation [3].

### C. Example: Superdense Coding

The superdense coding example presented by Schumacher and Westmoreland [1] can be directly implemented in  $\mathcal{R}II$ . Alice and Bob initially share an entangled state, represented by  $\{\{F, F\}\} \uplus \{\{T, T\}\}$ . Depending on Alice's choice of sending numbers 0 to 3, Alice applies one of the following operations on the first bit:

$$\begin{aligned} \text{alice}_0 &= \text{arr } id \\ \text{alice}_1 &= \text{arr } (\oplus \circ \otimes \times \circ \oplus) \\ \text{alice}_2 &= s_2 r \ (state \ (\{\{F, F\}\} \uplus \{\{T, F\}\} \uplus \{\{T, T\}\})) \\ \text{alice}_3 &= \text{alice}_1 \ggg \text{alice}_2 \end{aligned}$$

By measuring in a particular dual basis described below, Bob will deterministically set a different dual vector for each possible operation that Alice could have performed. The entire example is then written as follows:

$$\text{measure } ((\text{first } \text{alice}_n) \overline{\otimes} (\{\{F, F\}\} \uplus \{\{T, T\}\})) \ \text{dualbasis}$$

$$\text{where } \text{dualbasis} = \{ \begin{aligned} &\{\{F, T\}\} \uplus \{\{T, F\}\} \uplus \{\{T, T\}\}, \\ &\{\{F, F\}\} \uplus \{\{T, F\}\} \uplus \{\{T, T\}\}, \\ &\{\{F, T\}\} \uplus \{\{T, F\}\}, \\ &\{\{F, F\}\} \uplus \{\{T, T\}\} \end{aligned} }$$

Interestingly, it is possible to write this same example in conventional relational programming. The fact that intermediate results are accumulated using the standard union instead of the exclusive union is apparent in the results. However because this computation does not use significant intermediate steps, the only values that need to be treated specially are the final results. Indeed if the results that appear an even number of times are removed, then the Prolog execution performs the superdense coding exactly.

The complete program and its execution are below:

<pre> r(false,false). r(true,true).  s(false,true). s(true,false).  u(false,false). u(false,true). u(true,true).  v(false,false). v(false,true). v(true,false).  rd(false,true). rd(true,false). rd(true,true).  sd(false,false). sd(true,false). sd(true,true).  ud(false,true). ud(true,false). </pre>	<pre> vd(false,false). vd(true,true).  eq(false,false). eq(true,true).  id(false,false). id(true,true).  g(false,true). g(true,false).  k(false,false). k(true,false). k(true,true).  gk(X,Y) :- g(X,Z),k(Z,Y).  alice(0,X,Y) :- id(X,Y). alice(1,X,Y) :- g(X,Y). alice(2,X,Y) :- k(X,Y). alice(3,X,Y) :- gk(X,Y).  sdccoding(N,M) :- r(X,Y),alice(N,X,B),measure((B,Y),M).  measure((S1,S2),0) :- rd(B1,B2),dotP((B1,B2),(S1,S2)). measure((S1,S2),1) :- sd(B1,B2),dotP((B1,B2),(S1,S2)). measure((S1,S2),2) :- ud(B1,B2),dotP((B1,B2),(S1,S2)). measure((S1,S2),3) :- vd(B1,B2),dotP((B1,B2),(S1,S2)).  dotP((B1,B2),(S1,S2)) :- eq(B1,S1),eq(B2,S2). </pre>
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<pre> 28 ?- sdccoding(0,X). X = 1 ; X = 3 ; X = 0 ; X = 1 ; X = 3.  29 ?- sdccoding(1,X). X = 0 ; X = 1 ; X = 2 ; X = 0 ; X = 2.  30 ?- sdccoding(2,X). X = 1 ; </pre>	<pre> X = 3 ; X = 0 ; X = 2 ; X = 0 ; X = 1 ; X = 3.  31 ?- sdccoding(3,X). X = 1 ; X = 3 ; X = 0 ; X = 1 ; X = 2 ; X = 0 ; X = 2. </pre>
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## IX. CONCLUSION

We have distilled a programming model for the discrete quantum theory over the field of booleans recently introduced

by Schumacher and Westmoreland [1]. The model is expressed in a small calculus  $\mathcal{R}\text{II}$  with formal type rules and semantics. The language  $\mathcal{R}\text{II}$  is directly inspired by the computational structures of quantum mechanics previously identified by Abramsky and Coecke and Selinger. The semantics of  $\mathcal{R}\text{II}$  is sound with respect to both a relational model with exclusive unions or a vector space model over the field of booleans. Computationally  $\mathcal{R}\text{II}$  is a relational programming language surprisingly similar to traditional relational programming languages like Prolog with the significant difference that where Prolog would accumulate all possible answers with repetitions,  $\mathcal{R}\text{II}$  allows interference between possible answers.

A natural goal of our research is to unravel the mathematical underpinnings behind the power of quantum computation. By analyzing models such as those presented in this paper, we hope we will be able to shed some light on this fundamental issue. Deutsch's quantum algorithm, which establishes in a single measurement whether a boolean function is constant or balanced, makes use of interference and quantum parallelism. Although discrete quantum computing over the field of booleans,  $\text{DQC}_2$ , contains both these properties, Deutsch's algorithm cannot be efficiently implemented within this model. This example illustrates the non-trivial character of the quest to disentangle the power of quantum computation. In more detail, having eliminated much of the structure of actual quantum mechanics, the connection between  $\text{DQC}_2$  and conventional relational programming singles out the use of exclusive union as the source of all the "quantum-ness". Assuming that the exclusive union would be performed in "constant time" by a quantum computer, more work is needed to establish whether it is possible to write more efficient algorithms using the exclusive union.

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